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Bayesian Inference for the Reliability Parameter under the Inverse Rayleigh Distribution

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ABSTRACT

The parameter estimation problem of the probability $P(X > Y)$ for the inverse Rayleigh distribution is the main focus of this study. The maximum likelihood and Bayesian estimation methods are taken into consideration. Importance sampling and Lindley approximation techniques are used in Bayesian inference. The maximum likelihood approximation is used to generate asymptotic confidence intervals. The importance sampling approximation is also used to obtain Bayesian credible intervals. A simulation study is conducted to evaluate and compare the performance of the proposed estimation methods. The results indicate that Bayesian estimators perform better than maximum likelihood estimators in many cases. Moreover, Bayesian credible intervals are shorter than the asymptotic confidence intervals, especially for small sample sizes. Finally, a real-world application is conducted to improve the methods presented in this study.

Keywords: *Bayesian inference, Inverse Rayleigh distribution, Monte-Carlo simulation, System reliability*

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1. Introduction

The stress-strength model, a widely used framework in reliability analysis, defines system reliability as $R = P(X > Y)$. X and Y are independent random variables, representing the strength and stress, respectively. A system fails when the applied stress exceeds the components strength; otherwise, it functions successfully. This reliability measure, denoted by R, quantifies the probability that a system withstands the operational stress, making it central to classical stress-strength reliability theory. The subject has many applications in various areas, see [1]. Estimation of R has been extensively studied in the statistical literature. Previous studies have considered different distributions and estimation methods, see [2-8].

The inverse Rayleigh (IR) distribution was introduced by Trayer [9] to model survival and reliability data sets. After that, Voda [10] developed IR distribution. He studied its characteristics and the scale parameters of maximum likelihood estimation (MLE). Additionally, Gharraph [11] provided closed-form expressions for key statistical measures, including the mean, median, mode, and various types of averages. Moreover, Soliman [12] used Bayesian estimation and classical distribution methods to estimate the parameters. [13] considered the estimation of the probability density function and the cumulative distribution function on the IR distribution. Extensions of the IR distribution, such as the modified IR, half-logistic IR, Beta IR, and exponentiated IR distributions, have been proposed and applied in various contexts, see [14-18]. These studies demonstrate the versatility of the IR distribution in modeling right-skewed and lifetime data.

The main aim of this paper is to investigate Bayesian and MLE methods for $R = P(X > Y)$ where $X \sim IR(\theta)$ and $Y \sim IR(\alpha)$ they are independently distributed. In reliability and survival analysis, the MLE method is one of the most commonly used classical estimation. However, it is known that only when the sample size is large enough can the important characteristics of MLEs become available, see [19]. Survival and reliability studies are generally based on small sample sizes, in practice. In these situations, Bayesian parameter estimation methods have been proposed by many authors. For example, COVID-19 data with the Modified Fréchet–Exponential distribution were examined using approximate Bayesian estimators and MLEs by Akdam [20]. Similarly, Tanış et al. [21] compared the performance of approximate Bayesian estimators and classical estimators for the Log-Dagum distribution through a simulation study. Sharma et al. [22] examined Bayesian estimations using the Lindley approximation for the interval-censored Lindley distribution. Tanış et al. [23] compared both Bayesian and MLE methods by considering the $P(X > Y)$ estimation problem for the exponential power distribution. In addition, Dutta et al. [24] obtained parametric inference using Type II censored data based on a modified random variable model for simple stepped voltage lifetime testing.

These studies demonstrate the applicability of Bayesian estimation methods to different distributions and problem structures. This motivated us to study Bayesian methods in the context of the inverse Rayleigh distribution for $R = P(X > Y)$.

Building on these contributions, when compared with existing studies, the proposed method shows improved performance. First, the inverse Rayleigh distribution, which has been addressed to a limited extent in previous studies, is examined within the scope of stress-strength reliability analysis. Second, classical and Bayesian inference methods are compared, and MLE and Bayesian estimation techniques are examined within a comprehensive framework. Lastly, analytical and approximation techniques are used to construct Bayesian estimators under the squared error loss function (SELF). In these ways, the suggested method differs from previous research and expands the application areas for the analysis of reliability and lifetime data.

To the best of our knowledge, this study contributes to the limited literature on this topic. The reason for considering the inverse Rayleigh distribution in $R = P(X > Y)$ is that it is another model currently used for right-skewed data, similar to the inverse Weibull and gamma distributions.

The remainder of the paper is structured as follows: Section 2 introduces the IR distribution and its key properties. In Section 3, we derive the MLEs for the unknown parameters and examine their asymptotic properties. Bayesian estimators under the squared error loss function (SELF) are also constructed, utilizing both Lindley's approximation and importance sampling approximation methods. Section 4 presents a Monte Carlo simulation study conducted to evaluate and compare the performance of the proposed estimators. In Section 5, a real data example is analyzed. Finally, Section 6 concludes the paper with a summary of findings and suggestions for future work.

2. Inverse Rayleigh Distribution

The one- parameter IR distribution has the following probability density function (PDF) and cumulative distribution function (CDF)

$$f(x;\theta) = \frac{2\theta}{x^3} e^{-\left(\frac{\theta}{x^2}\right)} \quad \theta, x > 0, \tag{1}$$

$$F(x) = e^{-\left(\frac{\theta}{x^2}\right)} \quad \theta, x > 0, \tag{2}$$

respectively. Here ,θ is the scale parameter.

$$R = \frac{\theta}{\alpha + \theta} \tag{3}$$

The applications of the IR distribution and previous studies related to this distribution are presented in detail in section 1. This subsection aims to provide an understanding of the parameter estimation methods discussed in sections (3-4) by presenting the basic properties (PDF, and CDF) of the IR distribution. Furthermore, this Section provides a theoretical basis for the stress-strength reliability analysis examined in the study.

3. Maximum Likelihood Estimation

Let $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_m)$ be independent random samples from $X \sim IR(\theta)$ and $Y \sim IR(\alpha)$, respectively. Based on the observations X and Y the likelihood function of θ and α is:

$$L(\theta, \alpha | x, y) \propto (2\theta)^n \prod_{i=1}^{n} x_i^{-3} e^{-\theta \sum_{i=1}^{n} x_i^{-2}} (2\alpha)^m \prod_{j=1}^{m} y_j^{-3} e^{-\alpha \sum_{j=1}^{m} y_j^{-2}} \tag{4}$$

Then, the log-likelihood function is

$$lnL = nln2 + nln\theta - 3\sum_{i=1}^{n} lnx_i - \theta \sum_{i=1}^{n} x_i^{-2} + mln2 + mln\alpha - 3\sum_{j=1}^{m} lny_j - \alpha \sum_{j=1}^{m} y_j^{-2} \tag{5}$$

The following forms of the likelihood equations are obtained by taking the derivatives of (5) with regard to the θ and α parameters:

$$\frac{\partial lnL}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-2} = 0, \tag{6}$$

and

Now let, $X \sim IR(\theta)$ be independent of $Y \sim IR(\alpha)$. Then, $R = P(X > Y) = 1 - P(X \leq Y)$

$$= 1 - \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(x,y) dy dx$$

$$= 1 - \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{2\theta}{x^3} e^{-\left(\frac{\theta}{x^2}\right)} \frac{2\alpha}{y^3} e^{-\left(\frac{\alpha}{y^2}\right)} dy dx.$$

If we let $\alpha y^{-2} = u$ and $-2\alpha y^{-3} dy = du$.

$$= 1 - \int_{0}^{\infty} \frac{2\alpha}{x^3} e^{-\left(\frac{\alpha}{x^2}\right)} \int_{\theta/x^2}^{\infty} e^{-u} du dx.$$

Similarly, if we are take $x^{-2} = t$ and $-2x^{-3} dx = dt$,

$$= 1 - \int_{0}^{\infty} \frac{2\alpha}{x^3} e^{\frac{-(\alpha+\theta)}{x^2}} dx .$$

$$\frac{\partial lnL}{\partial \alpha} = \frac{m}{\alpha} - \sum_{j=1}^{m} y_j^{-2} = 0, \quad (7)$$

respectively. Then, by solving Equations (6)-(7), the MLEs of the parameters θ and α are obtained as follows:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i^{-2}} \quad \text{and } \hat{\alpha} = \frac{m}{\sum_{j=1}^{m} y_j^{-2}}. \quad (8)$$

By the invariance property of MLEs, see [25], substituting $\hat{\theta}$ and $\hat{\alpha}$ into Equation (3) yields the MLE of R as follows:

$$\hat{R} = \frac{\hat{\theta}}{\hat{\theta} + \hat{\alpha}}.$$

After that, we use asymptotic distributions of $\hat{R}$ to generate the asymptotic confidence interval of R. The asymptotic distributions of $\hat{\theta}$ and $\hat{\alpha}$ are obtained to derive the asymptotic distributions of $\hat{R}$.

The asymptotic distribution of the MLEs of the parameters θ and α is

Then, the approximate estimate of $Var(\hat{R})$ is

$$Var(\hat{R}) \cong Var(\hat{\theta}) \left(\frac{\partial \hat{R}}{\partial \theta}\right)^2 + Var(\hat{\alpha}) \left(\frac{\partial \hat{R}}{\partial \alpha}\right)^2.$$

Thus, $(\hat{R} - R)/\sqrt{Var(\hat{R})} \sim N(0,1)$, asymptotically. This result yields the asymptotic $100(1-\gamma)\%$ confidence interval R as

$$\hat{R} \pm Z_{\gamma/2} \sqrt{Var(\hat{R})}. \quad (11)$$

Algorithm 1: The algorithm of the MLE:

Step1: Generate independent samples from $X \sim IR(\theta)$ and $Y \sim IR(\alpha)$, respectively, as defined above.

Step 2: Construct the likelihood and log-likelihood functions from Equations (4) and (5), respectively.

Step 3: Differentiate the log-likelihood function with respect to θ and α equations (6)-(7) to obtain the likelihood equations.

Step 4: Solve the likelihood equations analytically to obtain the MLEs

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i^{-2}} \text{ and } \hat{\alpha} = \frac{m}{\sum_{j=1}^{m} y_j^{-2}}.$$

Step 5: Obtain the MLE of R, using the invariance property of the MLEs.

$$\begin{pmatrix} \hat{\theta} \\ \hat{\alpha} \end{pmatrix} \sim AN\left(\begin{pmatrix} \theta \\ \alpha \end{pmatrix}, I^{-1}(\theta, \alpha)\right). \quad (9)$$

Here, $I^{-1}(\theta, \alpha)$ is the inverse of the observed Fisher information matrix of the parameters θ and α.

$$I^{-1}(\theta,\alpha) \begin{bmatrix} -E\left(\frac{\partial^2 logL}{\partial^2 \theta}\right) & -E\left(\frac{\partial^2 logL}{\partial \theta \partial \alpha}\right) \\ -E\left(\frac{\partial^2 logL}{\partial \theta \partial \alpha}\right) & -E\left(\frac{\partial^2 logL}{\partial^2 \alpha}\right) \end{bmatrix}^{-1}_{(\theta,\alpha)=(\hat{\theta},\hat{\alpha})} = \begin{bmatrix} \hat{\sigma}_{11}, & \hat{\sigma}_{12} \\ \hat{\sigma}_{21}, & \hat{\sigma}_{22} \end{bmatrix}.$$

The elements of the Fisher information matrix $\left(I(\theta,\alpha) = [I_{ij}]_{2x2}\right)$ are given as follows:

$$\frac{\partial lnL}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-2} = 0, \quad (10)$$

order to find the approximate estimates of the variance of $\hat{R}$, we use delta method.

Let $G' = \left(\frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial \alpha}\right)$, where

$$\frac{\partial R}{\partial \theta} = \frac{\alpha}{(\alpha+\theta)^2} \quad \text{and} \quad \frac{\partial R}{\partial \alpha} = \frac{-\theta}{(\alpha+\theta)^2}.$$

$$\hat{R} = \frac{\hat{\theta}}{\hat{\theta}+\hat{\alpha}}$$

Step 6: Derive the asymptotic variance of $\hat{R}$ using the Fisher information matrix and the delta method.

Step 7: Construct the asymptotic $100(1-\gamma)\%$ confidence interval for R based on the asymptotic normality $\hat{R}$.

4. Bayesian Estimation

The Bayesian estimators the probability $P(X > Y)$ for the IR distribution under SELF are discussed in this section. In addition, Lindley and importance sampling approximation methods are considered to compute the Bayesian estimators.

In Bayesian parameter estimation, the choice of an appropriate loss function is very important. SELF is one of the most commonly used loss functions and has a symmetric structure. It is mathematically simple and analytically applicable, it is widely preferred in Bayesian estimation. These theoretical advantages have led to SELF being widely preferred in applications such as reliability analysis, Bayesian regression, and parameter estimation in complex models where uncertainty is critical. Recently, the SELF has been frequently used by many researchers for different lifetime distributions. Among them are [26-27].

Let, the parameters θ and α are assigned independent prior distributions, denoted by $Gamma(a_1, b_1)$ and $Gamma(a_2, b_2)$, respectively. Since the independent Gamma prior distribution is very flexible and suitable for our situation, it is reasonable to assume it.

Independent gamma priors on the parameters θ and α for lifespan distributions have been employed by numerous researchers, see, [28-29].

The proposed joint prior density function for θ and α is given as follows:

$$\pi(\theta,\alpha)=\pi(\theta)\pi(\alpha)\propto\theta^{a_1-1}e^{-b_1\theta}\alpha^{a_2-1}e^{-b_2\alpha}\quad,\qquad(12)$$

where$(a_1,\ b_1)$ and $(a_2,\ b_2)$are positive and known. The likelihood function is as follows based on the observations:

$$L(\theta,\alpha|x,y)\propto 2^{n+m}\theta^n\alpha^m\prod_{i=1}^{n}x_i^{-3}e^{-\theta\sum_{i=1}^{n}x_i^{-2}}\prod_{j=1}^{m}y_j^{-3}e^{-\alpha\sum_{j=1}^{m}y_j^{-2}}.\qquad(13)$$

By using (12) and (13), the joint posterior density function of θ and α is obtained as

$$\pi_1(\theta,\alpha|x,y)=\frac{L(x,y|\theta,\alpha)}{\int\int L(x,y|\theta,\alpha)\pi(\theta,\alpha)d\alpha d\theta}\propto\alpha^{m+a_2-1}\theta^{n+a_1-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{-2}-b_1\right)}\prod_{i=1}^{n}x_i^{-3}\,e^{-\alpha\left(\sum_{j=1}^{m}y_j^{-2}-b_2\right)}\prod_{j=1}^{m}y_j^{-3}.\qquad(14)$$

Then, the Bayesian estimation of R under SELF is obtained from the following expression

$$\hat{R}=E(R)=\int_0^\infty\int_0^\infty R\pi_1(\theta,\alpha|y,y)d\theta d\alpha.\qquad(15)$$

Since the integral provided in Equation (14) cannot be computed analytically, an alternative approximation is employed to obtain the Bayes estimator of R. *H*ere, Lindley's approximation and importance sampling approximations will be presented.

An approximation technique for calculating the ratio of the two integrals was presented by Lindley [30]. The posterior mean of an arbitrary function $u(\varphi)$ can be expressed using this method as follows:

$$E(u(\varphi)|x)=\frac{\int u(\varphi)e^{l(\varphi)+G(\varphi)}d\varphi}{\int e^{l(\varphi)+G(\varphi)}d\varphi},$$

where $u(\varphi)$ a function of $\varphi$is only, $l(\varphi)$is log-likelihood function and $G(\varphi)$ is log of the joint prior density function.

By using Lindley's method, $E(u(\varphi)|x)$ can be asymptotically approximated by:

4.1. Lindley's Approximation Method

$$E(u(\varphi)|x)\approx\left[u+0.5\sum_i\sum_j(u_{ij}+2u_i\rho_j)\sigma_{ij}+0.5\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right],\qquad(16)$$

where

$i;j;k;l=1,2,\dots n;\varphi=(\varphi_1,\varphi_2,\dots,\varphi_m)\ u_i=\frac{\partial u(\varphi)}{\partial\varphi_i},u_{ij}=\frac{\partial^2u(\varphi)}{\partial\varphi_i\partial\varphi_j},L_{ijk}=\frac{\partial^3u(\varphi)}{\partial\varphi_i\partial\varphi_j\partial\varphi_j},$

$\rho_j=\frac{\partial G(\varphi)}{\partial\varphi_i}$ and σ_{ij} are the elements of the covariance matrix.

In our case, to obtain the Bayes estimation of R using Lindley approximation, denoted as $\widehat{R}_L$, the following procedure is followed:

$$\widehat{R}_L=\hat{R}+0.5\left[\left(\hat{R}_{11}+2\hat{R}_1\hat{\rho}_1\right)\sigma_{11}+\left(\hat{R}_{22}+2\hat{R}_2\hat{\rho}_2\right)\sigma_{22}\right]+0.5\left[\hat{L}_{11}\hat{R}_1\sigma^2{}_{11}+\hat{L}_{22}\hat{R}_2\sigma^2{}_{22}\right],$$

where all the parameters are evaluated at $(\hat{\theta},\hat{\alpha})$. Also, $\rho_1=\frac{(a_1-1)}{\theta}-b_1,\ \rho_2=\frac{(a_2-1)}{\alpha}-b_2$ and,$L_{111}=\frac{-2n}{\theta^3},\ L_{222}=\frac{-2m}{\alpha^3},R_1=\frac{\partial R}{\partial\theta}=\frac{\alpha}{(\alpha+\theta)^2},\ R_{11}=\frac{\partial^2R}{\partial\theta^2}=\frac{-2\alpha}{(\alpha+\theta)^3}$,and, $R_2=\frac{\partial R}{\partial\alpha}=\frac{-\theta}{(\alpha+\theta)^2},R_{22}=\frac{\partial^2R}{\partial\alpha^2}=\frac{2\theta}{(\alpha+\theta)^3}$, σ_{11} and σ_{22} , are defined as Equation (10).

4.2. Importance sampling

We suggest using an importance sampling technique in this subsection to calculate the Bayes estimators. For detailed information on importance sampling see, [31]. Equation (14) provides the posterior density function with respect to the parameters of interest. The posterior conditional density functions of θ and α can be expressed using this equation as follows:

$$\pi_1^*(\theta|x) \propto \theta^{n+a_1-1} e^{-\theta\left(\sum_{i=1}^{n} x_i^{-2} - b_1\right)}, \quad \pi_2^*(\alpha|y) \propto \alpha^{m+a_2-1} e^{-\alpha\left(\sum_{j=1}^{m} y_j^{-2} - b_2\right)} \tag{17}$$

By using (14), the Bayes estimates of any function of θ andα can be calculated using the importance sampling approximation.

We use the following procedure:

Algorithm 2: The algorithm of the importance sampling

Step1: Generate $\theta^1 \sim Gamma(n + a_1, \sum_{i=1}^{n} x_i^{-2} - b_1)$ and $\left(\theta^{(1)}|\alpha^{(1)}\right) \sim Gamma\left(m + a_2, \sum_{j=1}^{m} y_j^{-2} - b_2\right)$.

Step 2: Repeat this procedure to obtain $(\theta_1, \alpha_1), \ldots, (\theta_N, \alpha_N)$.

Step 3: Now, the approximate

$$\hat{R} = E(R|data) = \frac{1}{N}\sum_{i=1}^{N} R^{(i)}.$$

5. Simulation Study

An extensive Monte Carlo simulation study is conducted to compare the performance of the ML and Bayesian estimators in the previous section with respect to mean and mean square errors (MSEs) for different sample sizes. Additionally, the average confidence lengths (ACL), the average Bayesian confidence lengths (BCL), and coverage probabilities (CPs), calculated for each sample are given. We considered the sample size $n = m = 30, 50, 100$ and the parameters $\theta = \alpha = 0.5, 1, 2, 3$. In the Bayes case, two different priors were used, for the first case, $a_1 = b_1 = a_2 = b_2 = 0$ were taken and say them prior-I. For the second case, the true values of the parameters were selected as the prior mean. In other words, $\alpha = a_1/b_1$ and $\theta = a_2/b_2$ are taken and say them prior- II. Both Lindley and importance sampling approximation methods were used to determine the Bayes estimators of the R under SELF with these priors. For the computation, 10,000 importance sampling were considered. Matlab R 2013 was used for all calculations, with 10,000 replications for each case. The mean and mean square error (MSE) values are given in Table 1. The ACL, BCL, and CP values are given in Table 2.

Table 1. The Mean and MSE values of R

R	(θ, α)	$n = m$	MLE		Lindley		Importance		Lindley		Importance	
			Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
0.2000	(0.5,2)	30	0.2089	0.0015	0.2120	0.0016	0.2106	0.0014	0.2065	0.0014	0.2054	0.0009
		50	0.2005	0.0010	0.2024	0.0009	0.2001	0.0011	0.2017	0.0008	0.2010	0.0007
		100	0.1981	0.0006	0.1990	0.0006	0.1970	0.0005	0.2012	0.0005	0.2013	0.0004
0.6667	(2,1)	30	0.6704	0.0030	0.6680	0.0029	0.6599	0.0027	0.6565	0.0027	0.6616	0.0020
		50	0.6630	0.0020	0.6616	0.0020	0.6558	0.0021	0.6589	0.0020	0.6608	0.0016
		100	0.6631	0.0011	0.6624	0.0010	0.6552	0.0009	0.6643	0.0009	0.6624	0.0006
0.6000	(3,2)	30	0.5914	0.0037	0.5900	0.0036	0.5921	0.0033	0.6590	0.0026	0.6623	0.0022
		50	0.6002	0.0019	0.5993	0.0017	0.5973	0.0015	0.6015	0.0020	0.6017	0.0017
		100	0.6015	0.0013	0.6010	0.0011	0.5990	0.0012	0.6010	0.0015	0.6009	0.0008
0.5000	(1,1)	30	0.4981	0.0045	0.4979	0.0043	0.4987	0.0037	0.4927	0.0036	0.4932	0.0028
		50	0.4968	0.0017	0.4975	0.0014	0.4987	0.0010	0.4932	0.0027	0.4944	0.0020
		100	0.4996	0.0023	0.4995	0.0021	0.4998	0.0018	0.5021	0.0010	0.4987	0.0005
0.8000	(2,0.5)	30	0.7977	0.0013	0.7966	0.0014	0.7985	0.0018	0.7958	0.0011	0.7965	0.0012
		50	0.7974	0.0015	0.7978	0.0012	0.7990	0.0010	0.7966	0.0009	0.7968	0.0008
		100	0.7976	0.0006	0.7985	0.0005	0.7984	0.0007	0.7970	0.0006	0.7980	0.0004
0.3333	(1,2)	30	0.3321	0.0034	0.3344	0.0030	0.3340	0.0029	0.3340	0.0030	0.3343	0.0028
		50	0.3324	0.0018	0.3338	0.0017	0.3337	0.0015	0.3338	0.0023	0.3339	0.0027
		100	0.3358	0.0010	0.3366	0.0010	0.3327	0.0009	0.3330	0.0008	0.3338	0.0004
0.4000	(2,3)	30	0.3981	0.0039	0.3986	0.0038	0.3985	0.0037	0.4087	0.0031	0.3947	0.0029
		50	0.3986	0.0024	0.3996	0.0020	0.3993	0.0017	0.3969	0.0023	0.3981	0.0018
		100	0.4014	0.0012	0.3996	0.0008	0.4002	0.0007	0.4069	0.0012	0.4026	0.0009

Table 2. ACL and CP of % 95 confidence/ credible interval of R

R	(θ,α)	$n=m$	MLE		Prior-I		Prior-II	
			ACL	CP	BCL	CP	BCL	CP
0.2000	(0.5,2)	30	0.1953	0.9174	0.1780	0.9257	0.1685	0.9267
		50	0.1561	0.9353	0.1410	0.9401	0.1388	0.9488
		100	0.0978	0.9458	0.0859	0.9507	0.0787	0.9584
0.6667	(2,1)	30	0.2457	0.9102	0.2286	0.9156	0.2177	0.9202
		50	0.1766	0.9343	0.1609	0.9360	0.1556	0.9400
		100	0.1221	0.9450	0.1177	0.9489	0.1059	0.9505
0.6000	(3,2)	30	0.2516	0.9006	0.2348	0.9055	0.2400	0.9105
		50	0.1891	0.9312	0.1754	0.9389	0.1672	0.9280
		100	0.1366	0.9500	0.1345	0.9555	0.1296	0.9504
0.5000	(1,1)	30	0.2633	0.9152	0.2466	0.9155	0.2341	0.9187
		50	0.1956	0.9315	0.1878	0.9300	0.1756	0.9346
		100	0.1388	0.9481	0.1314	0.9504	0.1265	0.9521
0.8000	(2,0.5)	30	0.1662	0.9001	0.1444	0.9077	0.1398	0.9102
		50	0.1323	0.9288	0.1280	0.9301	0.1186	0.9326
		100	0.1052	0.9456	0.1009	0.9596	0.0987	0.9612
0.3333	(1,2)	30	0.2380	0.9334	0.2187	0.9320	0.2096	0.9337
		50	0.1811	0.9474	0.1806	0.9402	0.1759	0.9459
		100	0.1506	0.9572	0.1496	0.9604	0.1303	0.9616
0.4000	(2,3)	30	0.2575	0.9116	0.2496	0.9202	0.2412	0.9152
		50	0.2151	0.9317	0.1929	0.9410	0.1806	09397
		100	0.1861	0.9505	0.1602	0.9555	0.1584	0.9594

From the simulation results presented in Tables 1 and 2, the following conclusions are presented

- First, as the sample size increases, both the mean and the MSE values of all estimators decrease, indicating that the estimators are asymptotically consistent.
- When comparing Bayesian approximations with MLE, Bayesian approximations performs better than MLE in terms of means and MSE, especially for small sample sizes of all cases (Prior-I and Prior).
- Similarly, when comparing the Lindley and importance methods, the importance sampling approximation shows slightly better performance than the Lindley approximation in terms of mean and MSE in most cases.
- Considering the Prior-I and Prior-II, Prior-II generally has smaller bias and MSE values.
- In terms of interval estimation, the ACLs are generally wider than the BCLs under both prior distributions. Additionally, the CPs of BCLs are generally closer to the nominal 95% than those of ACLs.
- Overall, these findings suggest that while all methods provide reasonable point estimators, the accuracy of the estimates depends on both the sample size and the estimation method used.

6. Real Data Set

In this section, we present an application of the proposed methods using a real data set taken from the literature. The dataset, originally reported by Bain and Engelhart [32] involves representing the failure times (in hours) of the air conditioning systems of different airplanes. [33] demonstrated that the IR distribution fits quite well for airplane data sets. This data set is as follows:

Plane 7911 (X): 33, 47, 55, 56, 104, 176, 182, 220, 239, 246, 320.

Plane 720 (Y): 12, 21, 26, 27, 29, 29, 48, 57, 59, 70, 74, 153, 326, 386, 502

The estimation of the R using MLE, Lindley, and importance sampling approximation methods under Prior-I is given in Table 3 for the data set. In addition to the parameter estimations, the asymptotic confidence intervals (ACIs) and Bayesian confidence intervals (BCIs) are presented in Table 3.

Table 3. MLE and Bayesian estimation and interval estimation for airplanes data set.

MLE	Lindley	Importance	ACI	BCI
0.8390	0.8236	0.8340	(0.7498 0.9385)	(0.7564 0.9216)

The Bayes estimators are preferable to the ML estimators based on the MSE and mean criterion, as demonstrated by the simulation results shown in Tables 1 and 2.

7. Conclusion

This study evaluated the probability $R = P(X > Y)$ for IR distribution using classical and Bayesian parameter estimation. The MLE method was considered the classical approximation, while the Lindley approximation and importance methods were used for Bayesian estimation. Additionally, ACLs and BCLs were constructed for interval estimation.

Several important observations were obtained from the simulation study. First, Bayesian estimation methods generally performed better than the MLE method, especially for small sizes $(n = 30)$. Among Bayesian approximations, the importance sampling method generally provided slightly more accurate estimates than Lindley's approximation. In interval estimations, the BCIs generally have a narrower range than ACLs. Furthermore, the CPs of BCLs are generally closer to the nominal 95% level compared to ACLs.

Our results are consistent with previous studies in the literature. Bayesian methods generally provide more reliable estimates, especially for small sample sizes or when prior information is available. Furthermore, this study demonstrates that right-skewed lifetime data can improve decision-making in engineering, maintenance planning, and risk assessment fields where such data are frequently encountered. The findings offer practical applications in the analysis of reliability and lifetime data.

For future research, potential extensions include exploring alternative prior distributions, applying the proposed methods to more complex or multivariate inverse Rayleigh models, and conducting studies with smaller sample sizes ($n = 5, 10, 20$). Such investigations would offer deeper insights into Bayesian estimation methods and expand their applicability in reliability and lifetime data analysis.

Conflicts of interest

The author declare that there is no conflict of interest in this respect.

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