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Numerical solution of the controlled Duffing oscillator by semi-orthogonal spline wavelets

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This paper presents a numerical method for solving the controlled Duffing oscillator. The method can be extended to nonlinear calculus of variations and optimal control problems. The method is based upon compactly supported linear semi-orthogonal B-spline wavelets. The differential and integral expressions which arise in the system dynamics, the performance index and the boundary conditions are converted into some algebraic equations which can be solved for the unknown coefficients. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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1. Introduction

The classical Duffing's equation was first introduced to study electronics and was published by Duffing in 1918 [1]. It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. The Duffing equation in one form or another has wide applications in signal processing [2], propagation of extremely short electromagnetic pulses in a nonlinear medium [3, 4], brain modelling [5], fuzzy modelling and adaptive control of uncertain chaotic systems [6, 7].

Vlassenbroeck and Van Dooren [8] introduced a direct method for the controlled Duffing oscillator. The method in [8] requires that the state and the control variables, the system dynamics, and the boundary conditions be expanded in Chebyshev series of order m with unknown coefficients. For the approximation of the integral in the performance index, a summation of order m_1 was used, and for the computation of the integrand in the performance index, a summation of order $N > m_1$ was employed. As a result, a rather complicated system of nonlinear equations have to be solved, for the unknowns, by some kind of iterative method. El-kady and Elbarbary [9] used Chebyshev polynomials for solving controlled Duffing oscillator. In [9], the control and state variables are approximated by Chebyshev series of different orders. The system dynamics, boundary conditions and performance index are approximated by using an explicit formula for the Chebyshev polynomials in terms of arbitrary order of their derivatives and a large system of nonlinear equations have to be solved.

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; in particular, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [10]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [11]. Wavelets can be separated into two distinct types, orthogonal and semi-orthogonal (SO) [12]. Publications on integral equation methods have shown a marked preference for orthogonal wavelets [13]. This is probably because the original wavelets, which were widely used for signal processing, were primarily orthogonal. In signal processing applications, unlike integral equation methods, the wavelet itself is never constructed since only its scaling function and coefficients are needed. However, orthogonal wavelets either have infinite support or a nonsymmetric and in some cases, fractal nature. These properties can make them a poor choice for characterization

of a function. In contrast, the SO wavelets have finite support, both even and odd symmetry and simple analytical expressions, ideal attributes of a basis function [13].

In this paper, we introduce an alternative computational method for solving the controlled Duffing oscillator. This method consists of reducing the controlled Duffing oscillator problem to a set of algebraic equations by using compactly supported linear SO B-spline wavelets, specially constructed for the bounded interval. The use of these wavelets is justified by their interesting properties. Among them the following can be explicitly cited [14]: they satisfy all the properties on a bounded interval that are verified by the usual wavelets on the real line, but they do not determine the difficulty related to enforce boundary conditions explicitly, when applying such wavelets to problems in finite bounded domain; unlike most of the continuous orthogonal wavelets, the SO compactly supported spline wavelets have closed form expressions.

The paper is organized as follows: in section 2, we describe the basic formulation of the B-spline scaling functions and wavelets on [0, 1] required for our subsequent development. Section 3 is devoted to the formulation of the controlled linear oscillator whose exact solution can be obtained by using Pontryagin's maximum principle [15]. Section 4 summarizes the application of this method to the controlled linear oscillator. In section 5, we apply the proposed method to the controlled Duffing oscillator, and in section 6, we report our numerical findings and demonstrate the accuracy of the proposed method.

2. B-spline scaling functions and wavelets on [0, 1]

When SO wavelets are constructed from B-splines of order m, the lowest octave level $j = j_0$ is determined by [16]

$$2^{j_0} \geqslant 2m - 1 \tag{1}$$

so as to give a minimum of one complete wavelet on the interval [0, 1]. In this paper, we will use a wavelet generated by a linear spline—the second-order cardinal B-spline basis function. From equation (1), the second-order B-spline of lowest level, which must be an integer, is determined to be $j_0 = 2$. This constrains all octave levels to $j \geqslant 2$.

As is the case with all SO wavelets, the second-order B-splines also serve as scaling functions. The second-order B-splines/scaling functions are given by

$$\phi_{j,k}(x) = \begin{cases} x_j - k, & k \leqslant x_j \leqslant k+1, \\ 2 - (x_j - k), & k+1 \leqslant x_j \leqslant k+2, \\ 0, & \text{otherwise}, \end{cases} \qquad k = 0, \ \ldots, \ 2^j - 2, \tag{2}$$

with the respective left- and right-hand side boundary scaling functions

$$\phi_{j,k}(x) = \begin{cases} 2 - (x_j - k), & 0 \leqslant x_j \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \qquad k = -1, \tag{3}$$

and

$$\phi_{j,k}(x) = \begin{cases} x_j - k, & k \leqslant x_j \leqslant k+1, \\ 0, & \text{otherwise}. \end{cases} \qquad k = 2^j - 1, \tag{4}$$

The actual coordinate position x is related to x_j, according to $x_j = 2^j x$. The second-order B-spline wavelets are given by

$$\psi_{j,k}(x) = \frac{1}{6} \begin{cases} x_j - k, & k \leqslant x_j \leqslant k+1/2, \\ 4 - 7(x_j - k), & k+1/2 \leqslant x_j \leqslant k+1, \\ -19 + 16(x_j - k), & k+1 \leqslant x_j \leqslant k+3/2, \\ 29 - 16(x_j - k), & k+3/2 \leqslant x_j \leqslant k+2, \\ -17 + 7(x_j - k), & k+2 \leqslant x_j \leqslant k+5/2, \\ 3 - (x_j - k), & k+5/2 \leqslant x_j \leqslant k+3, \\ 0, & \text{otherwise}, \end{cases} \qquad k = 0, \ \ldots, \ 2^j - 3, \tag{5}$$

with the respective left- and right-hand side boundary wavelets

$$\psi_{j,k}(x) = \frac{1}{6} \begin{cases} -6 + 23x_j, & 0 \leqslant x_j \leqslant 1/2, \\ 14 - 17x_j, & 1/2 \leqslant x_j \leqslant 1, \\ -10 + 7x_j, & 1 \leqslant x_j \leqslant 3/2, \\ 2 - x_j, & 3/2 \leqslant x_j \leqslant 2, \\ 0, & \text{otherwise}, \end{cases} \qquad k = -1, \tag{6}$$

and

$$\psi_{j,k}(x) = \frac{1}{6} \begin{cases} 2 - (k+2-x_j), & k \leqslant x_j \leqslant k+1/2, \\ -10 + 7(k+2-x_j), & k+1/2 \leqslant x_j \leqslant k+1, \\ 14 - 17(k+2-x_j), & k+1 \leqslant x_j \leqslant k+3/2, \\ -6 + 23(k+2-x_j), & k+3/2 \leqslant x_j \leqslant k+2, \\ 0, & \text{otherwise}. \end{cases} \qquad k = 2^j - 2 \tag{7}$$

For example, for $j = 2$ the inner scaling functions are obtained by putting $k = 0, 1, 2$ in equation (4) as

$$\phi_{2,0}(x) = \begin{cases} 4x, & 0 \leqslant x < 1/4, \\ 2 - 4x, & 1/4 \leqslant x < 1/2, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

$$\phi_{2,1}(x) = \begin{cases} 4x - 1, & 1/4 \leqslant x \leqslant 1/2, \\ 1 - 4x, & 1/2 \leqslant x \leqslant 3/4, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

and

$$\phi_{2,2}(x) = \begin{cases} 4x - 2, & 1/2 \leqslant x \leqslant 3/4, \\ -4x, & 3/4 \leqslant x \leqslant 1, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Also, for $j = 2$ the left- and right-hand side boundary scaling functions are obtained by putting $j = 2$, $k = -1$ and $k = 3$ in equations (3) and (4) respectively as

$$\phi_{2,-1}(x) = \begin{cases} 1 - 4x, & 0 \leqslant x \leqslant 1/4, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

and

$$\phi_{2,3}(x) = \begin{cases} 4x - 3, & 3/4 \leqslant x \leqslant 1, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Similarly, for $j = 2$ the inner wavelet functions are obtained by putting $j = 2$, $k = 0$ and $k = 1$ in equation (5) and the left- and right-hand side boundary wavelets are obtained by putting $j = 2$, $k = -1$ and $k = 2$ in equations (6) and (7) respectively.

3. Function approximation

A function $f(x)$ defined over [0, 1] may be represented by B-spline wavelets as

$$f(x) = \sum_{k=-1}^{3} c_k \phi_{2,k} + \sum_{i=2}^{\infty} \sum_{j=-1}^{2^{(i-1)}} \mathrm{d}_{i,j} \psi_{i,j}, \tag{13}$$

where $\phi_{2,k}$ and $\psi_{i,j}$ are scaling and wavelets functions respectively. If the infinite series in equation (13) is truncated, then equation (13) can be written as

$$f(x) = \sum_{k=-1}^{3} c_k \phi_{2,k} + \sum_{i=2}^{M} \sum_{j=-1}^{2^{(i-1)}} \mathrm{d}_{i,j} \psi_{i,j} = \alpha^T \Psi, \tag{14}$$

where α and Ψ are $(2^{(M+1)} + 1) \times 1$ vectors given by

$$\alpha = [c_{-1}, c_0, \ \ldots, \ c_3, d_{2,-1}, \ \ldots, \ d_{2,2}, d_{3,-1}, \ \ldots, \ d_{3,6}, \ \ldots, \ d_{M,-1}, \ \ldots, \ d_{M,2^{(M-1)}}]^T, \tag{15}$$

$$\Psi = [\phi_{2,-1}, \phi_{2,0}, \ \ldots, \ \phi_{2,3}, \psi_{2,-1}, \ \ldots, \ \psi_{2,2}, \psi_{3,-1}, \ \ldots, \ \psi_{3,6}, \ \ldots, \ \psi_{M,-1}, \ \ldots, \ \psi_{M,2^{(M-1)}}]^T. \tag{16}$$

Let

$$\Phi = [\phi_{2,-1}(x), \phi_{2,0}(x), \phi_{2,1}(x), \phi_{2,2}(x), \phi_{2,3}(x)]^T, \tag{17}$$

$$\Omega = [\psi_{2,-1}, \ \ldots, \ \psi_{2,2}, \psi_{3,-1}, \ \ldots, \ \psi_{3,6}, \ \ldots, \ \psi_{M,-1}, \ \ldots, \ \psi_{M,2^{(M-1)}}]^T. \tag{18}$$

Using equations (8)–(12) and equation (17) we get

$$\int_0^1 \Phi \Phi^T \mathrm{d}x = P_1 = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \tag{19}$$

and from equations (5)–(7) and equation (18), we have

$$\int_0^1 \Omega \Omega^T \mathrm{d}x = P_2 = \begin{bmatrix} N_{4\times 4} & & & \\ & \frac{1}{2} N_{8\times 8} & & \\ & & \ddots & \\ & & & \frac{1}{2^{M-2}} N_{2^M \times 2^M} \end{bmatrix}, \tag{20}$$

where N is a five diagonal matrix given by

$$N = \begin{bmatrix} \frac{2}{27} & \frac{1}{96} & -\frac{-1}{864} & 0 & 0 & \cdots & 0 \\ \frac{1}{96} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & 0 & \cdots & 0 \\ -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} \\ 0 & \cdots & 0 & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} \\ 0 & \cdots & 0 & 0 & -\frac{1}{864} & \frac{1}{96} & \frac{2}{27} \end{bmatrix}.$$

4. The controlled linear oscillator

Consider the optimal control problem of a linear oscillator given in [9] as

$$J = \frac{1}{2} \int_{-T}^{0} u^2(t) \, \mathrm{d}t, \tag{21}$$

subject to

$$\ddot{x}(t) + \omega^2 x(t) = u(t), \tag{22}$$

with

$$x(-T) = x_1, \quad x(0) = 0, \quad \dot{x}(-T) = x_2, \quad \dot{x}(0) = 0, \tag{23}$$

where $-T \leqslant t \leqslant 0$ and T is known. Equations (22) and (23) are equivalent to

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = -\omega^2 x_1(t) + u(t), \tag{24}$$

$$x_1(-T) = x_1, \qquad x_2(-T) = x_2, \tag{25}$$

$$x_1(0) = 0, \qquad x_2(0) = 0. \tag{26}$$

The problem is to find the control vector $u(t)$ which minimizes equation (21) subject to equations (24)–(26). The exact solution of the controlled linear oscillator can be obtained by applying Ponteryagin's maximum principle [15]

$$\begin{aligned} x_1(t) &= \frac{1}{2\omega^2}[A_1 \omega t \sin \omega t + A_2(\sin \omega t - \omega t \cos \omega t)], \\ x_2(t) &= \frac{1}{2\omega}[A_1(\sin \omega t + \omega t \cos \omega t) + A_2 \omega t \sin \omega t], \\ u(t) &= A_1 \cos \omega t + A_2 \sin \omega t, \\ J &= \frac{1}{8\omega}[2\omega T(A_1^2 + A_2^2) + (A_1^2 - A_2^2) \sin 2\omega T \\ &\quad - 4A_1 A_2 \sin^2 \omega T], \end{aligned}$$

where

$$A_1 = \frac{2\omega[x_1 \omega^2 T \ \sin \omega T - x_2(\omega T \ \cos \omega T - \sin \omega T)]}{\omega^2 T^2 - \sin^2 \omega T},$$

$$A_2 = \frac{2\omega^2[x_2 T \ \sin \omega T + x_1(\sin \omega T + \omega T + \omega T \ \cos \omega T)]}{\omega^2 T^2 - \sin^2 \omega T}.$$

5. Approximation of the controlled linear oscillator

In order to use the linear B-spline wavelets, we introduce the transformation $t = T(\tau - 1)$. The optimal control problem can be restated as follows

$$J = \frac{T}{2} \int_0^1 u^2(\tau) \, \mathrm{d}\tau, \tag{27}$$

subject to

$$\ddot{x}(\tau) = T^2[-\omega^2 x(\tau) + u(\tau)], \quad 0 \leqslant \tau \leqslant 1, \tag{28}$$

with

$$x(0) = x_1, \quad x(1) = 0, \tag{29}$$

$$\dot{x}(0) = T x_2, \quad \dot{x}(1) = 0. \tag{30}$$

In view of equations (14)–(16), $\ddot{x}(\tau)$ and $u(\tau)$ may be represented by

$$\ddot{x}(\tau) = D^T \Psi(\tau), \tag{31}$$

$$u(\tau) = E^T \Psi(\tau), \tag{32}$$

where D and E are unknown $(2^{M+1}+1)\times 1$ vectors. By integrating equation (31) and using equations (29) and (30), we get

$$\dot{x}(\tau) = D^T H(\tau) + T x_2, \tag{33}$$

$$x(\tau) = D^T G(\tau) + \tau T x_2 + x_1, \tag{34}$$

where

$$H(\tau) = \int_0^\tau \Psi(\theta)\,\mathrm{d}\theta, \quad G(\tau) = \int_0^\tau H(\theta)\,\mathrm{d}\theta.$$

By substituting equations (31), (32) and (34) in equation (28), we get

$$D^T \Psi(\tau) = T^2[-\omega^2(D^T G(\tau) + \tau T x_2 + x_1) + E^T \Psi(\tau)], \quad 0 \leqslant \tau \leqslant 1, \tag{35}$$

A collocation scheme is defined in equation (35) by evaluating the result at the points $\tau_i = i - 1/2^{M+1}$, $i = 1, 2, \ldots, 2^{M+1}+1$, that generate a set of $2^{(M+1)}+1$ algebraic equations as

$$A_i^* = D^T \Psi(\tau_i) - T^2[-\omega^2(D^T G(\tau_i) + \tau_i T x_2 + x_1) + E^T \Psi(\tau_i)] = 0, \quad i = 1, \ldots, 2^{M+1}+1. \tag{36}$$

From equations (29), (30), (33) and (34), we get

$$x(1) = D^T G(1) + T x_2 + x_1 = 0, \tag{37}$$

$$\dot{x}(1) = D^T H(1) + T x_2 = 0. \tag{38}$$

Substituting equation (32) in equation (27), and using equations (19) and (20), we get

$$J = \frac{T}{2}\int_0^1 [E^T \Psi(\tau)][\Psi^T(\tau) E]\,\mathrm{d}\tau = \frac{T}{2} E^T \int_0^1 [\Psi(\tau)\Psi^T(\tau)]\,\mathrm{d}\tau\, E = \frac{T}{2} E^T P E, \tag{39}$$

where

$$P = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}.$$

The minimization problem of equation (27) subject to equations (28)–(30) reduces to a parameter optimization problem which can be stated as follows. Find D and E so that equation (39) is minimized subject to the constrains in equations (36)–(38). Let

$$J^*(D, E, \lambda_1, \ldots, \lambda_{2^{M+1}+1}, \mu_1.\mu_2) = J + \sum_{i=1}^{2^{M+1}+1} \lambda_i A_i^* + \mu_1 x(1) + \mu_2 \dot{x}(1),$$

where the scalars $\lambda_i, i = 1, \ldots, 2^{M+1}+1$, μ_1 and μ_2 represent the unknown Lagrange multipliers. The necessary conditions for a minimum are

$$\frac{\partial J^*}{\partial D} = 0, \qquad \frac{\partial J^*}{\partial E} = 0. \tag{40}$$

Equation (40) together with equations (36)–(38) are linear algebraic equations which can be solved for unknowns D, E, $\lambda_i, i = 1, \ldots, 2^{M+1}+1, \mu_1$ and μ_2.

6. The controlled Duffing oscillator

The optimal control of Duffing oscillator is described by the nonlinear differential equation [9],

$$\ddot{x}(t) + \omega^2 x(t) + \epsilon x^3(t) = u(t), \quad -T \leqslant t \leqslant 0, \tag{41}$$

subject to same boundary condition as in linear case with the same performance index. The exact solution in this case is not known. By using the transformation $t = T(\tau - 1)$ and equations (31)–(34), we obtain the following approximation for equation (41).

$$D^T \Psi(\tau) = T^2[-\omega^2(D^T G(\tau) + \tau T x_2 + x_1) + (D^T G(\tau) + \tau T x_2 + x_1)^3 + E^T \Psi(\tau)], \quad 0 \leqslant \tau \leqslant 1. \tag{42}$$

A collocation scheme is defined in equation (42) by evaluating the result at the points $\tau_i = i - 1/2^{M+1}$, $i = 1, 2, \ldots, 2^{M+1}+1$, that generate a set of $2^{(M+1)}+1$ algebraic equations as

$$D^T \Psi(\tau_i) - T^2[-\omega^2(D^T G(\tau_i) + \tau_i T x_2 + x_1) + E^T \Psi(\tau_i)] = 0, \quad i = 1, \ldots, 2^{M+1}+1, \tag{43}$$

Equation (43) is a nonlinear equation, and hence the determining equations similar to equation (40) are also nonlinear equations which can be solved by Newton's iterative method. The starting values required to start Newton's iterative method can be chosen from the controlled linear oscillator i.e., $\epsilon = 0$, see [8].

7. The numerical results

We applied the present method to a controlled linear oscillator for various values of M with the following choice of the numerical values of the parameters in the standard case

$$\omega = 1, \qquad T = 2, \qquad x(-1) = 0.5, \qquad \dot{x}(-1) = -0.5$$

In table 1, we report the errors $e_x = \|x(\tau) - x^*(\tau)\|_2$ and $e_u = \|u(\tau) - u^*(\tau)\|_2$, where $x(\tau)$ and $u(\tau)$ are obtained from the present method, and $x^*(\tau)$ and $u^*(\tau)$ are exact solutions of the problem. The symbol $\|\cdot\|_2$ denotes the L^2 norm.

In table 2, a comparison is made between the present method for $M = 4, 6, 8$ and 10 together with the solution obtained by [8] for the controlled linear oscillator and the exact solution.

Table 1. Error's norm of estimated control and state functions.

	e_u	e_x
$M = 6$	3.4×10^{-5}	1.2×10^{-6}
$M = 8$	4.4×10^{-6}	1.0×10^{-7}
$M = 10$	5.5×10^{-7}	1.0×10^{-8}

Table 2. Estimated and exact values of J.

Methods	J
Method of [8]	
$m = 4, m_1 = 6, N = 12$	0.184917
$m = 7, m_1 = 10, N = 20$	0.18485854
$m = 10, m_1 = 15, N = 30$	0.1848585424
Present method	
$M = 4$	0.18485854
$M = 6$	0.1848585423
$M = 8$	0.18485854236
$M = 10$	0.1848585423638
Exact	0.1848585423638

Table 3. Error on J versus parameters.

Methods	Parameter	Error on J
Method of [8]	Standard case	2.7×10^{-16}
$(m = 10, m_1 = 15, N = 30)$	$w = 2$	9.7×10^{-11}
	$T = 3$	6.6×10^{-14}
	$x_1 = 1, x_2 = -1$	1.1×10^{-14}
Present method	Standard case	1.0×10^{-16}
$M = 8$	$w = 2$	5.2×10^{-16}
	$T = 3$	1.1×10^{-16}
	$x_1 = 1, x_2 = -1$	1.0×10^{-16}

Table 4. A comparison for the controlled Duffing oscillator J.

Methods	J
Method of [8]	
$m = 4, m_1 = 6, N = 12$	0.187531
$m = 7, m_1 = 10, N = 20$	0.18744484
$m = 10, m_1 = 15, N = 30$	0.1874448561
Present method	
$M = 6$	0.18744485613
$M = 8$	0.187444856135836
$M = 10$	0.187444856135836

Table 3 presents, for various values of the parameters w, T, x_1 and x_2, the maximum error using the present method for $M = 8$ on the performance index in comparison with the results obtained from the exact solution, together with the results obtained by the method of [8] with $m = 10$, $m_1 = 15$ and $N = 30$. Unlike the method in [8], using the present method, by increasing the value of some of the above parameters, and holding the other parameters fixed, the accuracy is not relatively lower than the accuracy obtained using the standard case.

In table 4, the approximation for the controlled Duffing oscillator using the present method for $M = 6, 8$ and 10 and the solutions obtained by the Chebyshev approximation [8] for the same numerical values of the parameters w, T, x_1 and x_2 as given in the standard case and where, in addition, the coefficient ε of the nonlinearity has been taken as $\varepsilon = 0.15$.

8. Conclusion

In the present work, a technique has been developed for obtaining the optimal solution of the controlled Duffing oscillator. The method is based upon compactly supported linear SO B-spline wavelets. The simplicity of these bases which simplifies the process of reducing nonlinear system dynamics to algebraic equations and the structure of the P matrix given in equation (39) makes this approach computationally very attractive.

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